On the Newtonian Limit in Gravity Models with Inverse Powers of R

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Abstract: I reconsider the problem of the Newtonian limit in nonlinear gravity models in the light of recently proposed models $\mathcal{L}_{grav} \sim \sqrt{-g}f(R)$ with inverse powers of R.
Expansion around a maximally symmetric local background with curvature scalar $R_0 > 0$ gives the correct Newtonian limit on length scales $\ll R_0^{-1/2}$ if the gravitational Lagrangian $\sqrt{-g}f(R)$ satisfies $|f(R_0)f''(R_0)| \ll 1$, and I propose two models with $f''(R_0) = 0$.

1 Introduction

The need for an effective or genuine cosmological constant to explain the faster than expected cosmological expansion in our epoch has generated a lot of activity on scalar field ("quintessence") models, where the potential energy or an otherwise anomalous dispersion relation of the quintessence accelerates the expansion.
On the other hand, it is known that curvature terms can also accelerate the expansion of the universe [1, 2, 3, 4, 5, 6], and the application of this mechanism to explain the current expansion rate has been denoted as curvature quintessence [7]. While this mechanism usually relies on higher order curvature terms, it has also been noticed recently that inclusion of an R^{-1} term in the gravitational Lagrangian would yield a scale factor $a(t) \propto t^2$ [7, 8, 9].
The model proposed recently by Carroll *et al.* (CDTT), $\mathcal{L} \sim R - (\mu^4/R)$ [9], fits into the framework of the so-called nonlinear gravity (NLG) models

$$\mathcal{L} = \frac{M^2}{2}\sqrt{-g}f(R) + \mathcal{L}_{matter}, \tag{1}$$

see [6, 10] and references there, and for brevity I denote models with $f(0) = \infty$ as *singular NLG models* in the sequel.
The generalized Einstein equations following from (1) are

$$\begin{aligned}\tilde{G}_{\mu\nu} &\equiv f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu f'(R) + g_{\mu\nu}\nabla^2 f'(R) \\ &= \frac{1}{M^2}T_{\mu\nu},\end{aligned} \tag{2}$$

and it is readily verified that $\nabla^\mu \tilde{G}_{\mu\nu} \equiv 0$. NLG theories usually assume[1] $f(R) = R + 6\ell^2 R^2 + \mathcal{O}(R^3)$, whence (2) admits flat Minkowski space as a maximally symmetric vacuum solution, and the Newtonian limit proceeds as in Einstein gravity, with additional Yukawa terms in the gravitational potential [12, 13, 14, 15]. Suppression of the Yukawa terms at macroscopic distances leaves only the the leading $1/r$ term, and one finds that $M = M_{Pl} \equiv (8\pi G_N)^{-1/2}$ is the reduced Planck mass as in Einstein gravity.

However, the model proposed in [9] does not allow flat Minkowski space as a solution, and the problem of the Newtonian limit is more intricate[2]. Intuitively one would expect that on length scales much smaller than an intrinsic curvature scale one should be able to recover the Newtonian limit, but intuition can be deceiving, and it is known in the framework of regular NLG models that these models may not admit a consistent weak field approximation. Therefore I propose the following approach to study this problem for singular NLG theories: Since our four-dimensional spacetime locally admits a ten-dimensional group of symmetry transformations, the Newtonian limit, if it exists, should be recoverable from expansion around a maximally symmetric local background geometry, which contrary to the regular case now will have to correspond to a curvature scalar $R_0 \neq 0$. This will be used in Sec. 3 to demonstrate that existence of a weak field approximation around a symmetric local background with Ricci scalar $R_0 > 0$ can be achieved by imposing the condition $f''(R_0) = 0$ on the singular NLG models. In these models M is then related to the reduced Planck mass through

$$M = M_{Pl} \Big/ \sqrt{f'(R_0)}\,.$$

However, before entering the discussion of the Newtonian limit in singular NLG models, I would like to revisit and slightly extend the evidence for accelerated expansion in these models in Sec. 2.

2 The cosmological behavior at late times

The cosmological evolution equations from (2) are quite complicated, but we can make a general statement about the late time expansion behavior of singular NLG models.

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[1] I follow the MTW conventions [11] for the signature of the metric and the definition of the Ricci tensor:

$$R_{\mu\nu} = R^\sigma{}_{\mu\sigma\nu} = \partial_\sigma \Gamma^\sigma{}_{\mu\nu} - \partial_\nu \Gamma^\sigma{}_{\mu\sigma} + \Gamma^\sigma{}_{\rho\sigma}\Gamma^\rho{}_{\mu\nu} - \Gamma^\sigma{}_{\rho\nu}\Gamma^\rho{}_{\mu\sigma}.$$

It is useful to keep that in mind when comparing with the literature on regular NLG models, because the relative signs between even and odd powers of R depend on these conventions.

[2] Capozziello *et al.* had noticed that $f(R) = R^{-1}$ yields $a(t) \propto t^2$ [7, 8], but did not further pursue this model. $f(R) = R^{-1}$ would not have a maximally symmetric vacuum solution.

Since the generalized Einstein equation (2) still implies energy-momentum conservation $\nabla^\mu T_{\mu\nu} = 0$, the time evolution of the scale factor $a(t)$ in a Friedmann model is still governed by the generalized Friedmann equation $\delta\mathcal{L}/\delta g^{00}|_{FRW\ metric} = 0$.
For the spatially flat FRW metric

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2$$

the generalized Friedmann equation following from (2) is

$$-3f'\left(6\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2}{a^2}\right)\frac{\ddot{a}}{a} + \frac{1}{2}f\left(6\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2}{a^2}\right) + 3\frac{\dot{a}}{a}\partial_0 f'\left(6\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2}{a^2}\right)$$
$$= \frac{1}{M^2}\varrho, \tag{3}$$

with $T_{00} = \varrho$.
In general this will be a third order equation for the scale factor. To analyze the late time behavior, we first assume that there is only ordinary dust and radiation in ϱ, whence the energy density can be neglected at late times for expanding solutions.
We then make a power law *ansatz* $a(t) \propto t^\alpha$, which yields

$$-3f'\left(\frac{6}{t^2}\alpha(2\alpha-1)\right)\frac{\alpha(\alpha-1)}{t^2} + \frac{1}{2}f\left(\frac{6}{t^2}\alpha(2\alpha-1)\right)$$
$$-36\frac{\alpha^2(2\alpha-1)}{t^4}f''\left(\frac{6}{t^2}\alpha(2\alpha-1)\right) = 0. \tag{4}$$

If R^{-n}, $n > 0$, is the leading order singularity in the singular NLG model $f(R)$, then at late times the contribution from this term will dominate all 3 terms in Eq. (4), with the same time dependence $\propto t^{2n}$. This yields the algebraic equation

$$n\frac{\alpha-1}{2(2\alpha-1)} + \frac{1}{2} - \frac{n(n+1)}{2\alpha-1} = 0,$$

which determines the expansion coefficient α:

$$\alpha = \frac{2n^2+3n+1}{n+2}. \tag{5}$$

This was found for $f(R) = R - \mu^{2n+2}R^{-n}$ in [9] through conformal transformation to a corresponding scalar quintessence model, and the corresponding result for $f(R) = R^n$, $n > 0$, is also spelled out in [16].
Note, however, that Eq. (3) is also compatible with exponential expansion $a(t) \propto \exp(Ht)$ at late times if $f(12H^2) = 6H^2 f'(12H^2)$ has a solution. Carroll *et al.* found in the metric formulation of their model that power law expansion is dynamically preferred [9]. Vollick looked at the Palatini formalism in the CDTT model and concluded that exponential expansion would arise in that formulation [17]. Our use of a symmetric local background geometry in the next section does not predicate the global late time expansion, but only assumes that spacetime should have maximal symmetry locally.

3 Expansions around maximally symmetric vacua

In the spirit of the philosophy outlined in Sec. 1 we now assume that the Newtonian limit should be recoverable through weak field expansion around a symmetric local background geometry: The maximally symmetric vacuum solutions satisfy

$$R_{\alpha\mu\beta\nu} = \frac{R_0}{12}(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\mu\beta}),$$

$$R_{\mu\nu} = \frac{R_0}{4}g_{\mu\nu},$$

where the constant curvature scalar R_0 satisfies

$$f'(R_0)R_0 = 2f(R_0). \tag{6}$$

In ordinary NLG theories this always permits $R_0 = 0$, but for singular NLG models this yields values $R_0 \neq 0$, and the challenge is to derive the Newtonian limit from the weak field expansion around the vacuum solution.
The first order expansion of Eq. (2) around a symmetric vacuum solution yields

$$\begin{aligned}&f'(R_0)\delta R_{\mu\nu} + \frac{1}{4}[f''(R_0)R_0 - 2f'(R_0)]g_{\mu\nu}\delta R\\ &-\frac{1}{2}f(R_0)\delta g_{\mu\nu} - f''(R_0)(\nabla_\mu\nabla_\nu\delta R - g_{\mu\nu}\nabla^2\delta R) = \frac{1}{M^2}T_{\mu\nu}\end{aligned} \tag{7}$$

or

$$\begin{aligned}&f'(R_0)\delta R_{\mu\nu} - \frac{1}{4}f''(R_0)R_0 g_{\mu\nu}\delta R - \frac{1}{2}f(R_0)\left(\delta g_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\delta g\right)\\ &-f''(R_0)\left(\nabla_\mu\nabla_\nu\delta R + \frac{1}{2}g_{\mu\nu}\nabla^2\delta R\right) = \frac{1}{M^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right).\end{aligned} \tag{8}$$

The first order variation of the Ricci tensor is

$$\begin{aligned}\delta R_{\mu\nu} &= \frac{1}{2}(\nabla_\mu\nabla^\sigma\delta g_{\sigma\nu} + \nabla_\nu\nabla^\sigma\delta g_{\sigma\mu} + \delta g_{\alpha\nu}R^\alpha{}_\mu + \delta g_{\alpha\mu}R^\alpha{}_\nu)\\ &\quad -\delta g_{\alpha\beta}R^\alpha{}_\mu{}^\beta{}_\nu - \frac{1}{2}\nabla^2\delta g_{\mu\nu} - \frac{1}{2}\nabla_\mu\nabla_\nu\delta g \qquad (9)\\ &= \frac{1}{2}(\nabla_\mu\nabla^\sigma\delta g_{\sigma\nu} + \nabla_\nu\nabla^\sigma\delta g_{\sigma\mu}) + \frac{1}{3}R_0\delta g_{\mu\nu} - \frac{1}{12}R_0 g_{\mu\nu}\delta g \qquad (10)\\ &\quad -\frac{1}{2}\nabla^2\delta g_{\mu\nu} - \frac{1}{2}\nabla_\mu\nabla_\nu\delta g,\end{aligned}$$

The mass term $f(R_0) \sim \mathcal{O}(R_0)$ vanishes in regular NLG theories, and should also be negligible in singular NLG theories at least up to length scales where Newton's law has been verified, which implies that $R_0 \sim \mu^2$ must correspond to a small mass scale μ. We also note that the mass and derivative terms following from Eqs. (8) and (10)

have the correct signs for non-oscillatory attractive solutions if $R_0 \geq 0$, $f'(R_0) > 0$ ($\Rightarrow f(R_0) \geq 0$), and $f''(R_0) \geq 0$.

In regular NLG theories $f(R_0) = f(0) = 0$, $f'(R_0) = 1$, and $f''(R_0) = 12\ell^2$ is assumed to be very small, such that the corresponding Yukawa terms are suppressed relative to the leading $1/r$ term at macroscopic distances. On the other hand, every set of observational tests of Newton's law can only cover a finite range of length scales. Therefore one might be tempted to conclude that very large ℓ is another possibility, such that e.g. the Yukawa term $\exp(-r/\ell)/r$ from $f(R) = R + 6\ell^2 R^2$ at observational distances also approximates a $1/r$ term and only rescales the ratio between M and M_{Pl} by a constant factor.

That this latter possibility is excluded in regular NLG theories was noticed already by Pechlaner and Sexl: $f''(0) = 12\ell^2$ has to be small for consistency of the weak field approximation, because otherwise domination of the fourth order terms would yield strong curvature on all length scales [12].

This reasoning carries over to the singular case, with minor modification: Due to the presence of a small mass term the Newtonian potential, if it exists in the theory, will always come from a limit of Yukawa terms. Yet we still have to confine the impact from the fourth order terms to small r. This will constrain the parameter space, because in singular NLG theories $f'(R_0) \sim \mathcal{O}(1)$, and $f''(R_0) \sim \mu^{-2}$ generically would imply that the fourth order derivative terms dominate the equation for $\delta g_{\mu\nu}$, thus spoiling the consistency of the weak field approximation.

The need for suppression of the fourth order terms can also be seen from the following simple example:

$$\Delta U(\mathbf{r}) - \mu^2 U(\mathbf{r}) - \frac{1}{2m^2}\Delta^2 U(\mathbf{r}) = \frac{1}{2M^2}\delta(\mathbf{r})$$

yields

$$U(\mathbf{r}) = -\frac{m}{8\pi M^2 r\sqrt{m^2 - 2\mu^2}}\left[\exp(-k_- r) - \exp(-k_+ r)\right],$$

with

$$k_\pm^2 = m^2 \pm m\sqrt{m^2 - 2\mu^2}.$$

This will give a Newtonian $1/r$ potential at distances $r \ll \mu^{-1}$ only if $\mu \ll m$:

$$\begin{aligned} U(\mathbf{r}) &\approx -\frac{1}{8\pi M^2 r}\left[\exp(-\mu r) - \exp(-\sqrt{2}mr)\right] \\ &\approx -\frac{1}{8\pi M^2 r}, \quad m^{-1} \ll r \ll \mu^{-1}. \end{aligned}$$

In the terminology of the singular NLG models this means that we need

$$|f(R_0) f''(R_0)| \ll 1,$$

while e.g. $f(R) = R - \mu^{2n+2}R^{-n}$ would yield $|f(R_0)f''(R_0)| = n(n+1)^2/(n+2)^2$. Therefore we either have to invoke a second small parameter in $f(R)$ such that *both* $f(R_0)$ and $f''(R_0)/r^4$ are small relative to $f'(R_0)/r^2$ at the length scales of interest. Or, since $f(R_0) \neq 0$ by Eq. (6), we arrange $f(R)$ such that the coefficient of μ^{-2} vanishes altogether, i.e. by choosing our model such that the solution of Eq. (6) satisfies

$$f''(R_0) = 0. \tag{11}$$

In that case the fourth order terms vanish in the weak field expansion and all the curvature contributions to Eq. (8) are subleading, such that for $r \ll \mu^{-1}$ a flat *ansatz* can be used to determine the local potential at these scales. In leading order this is then nothing but the ordinary Newtonian limit at these scales, with the only modification that

$$M = (8\pi G_N f'(R_0))^{-1/2} = M_{Pl} \Big/ \sqrt{f'(R_0)}\,. \tag{12}$$

4 Two simple examples of singular NLG models

4.1 The criterion (11) is satisfied e.g. by

$$\mathcal{L} = \frac{M^2}{2}\sqrt{-g}\left(R + \frac{R^2}{9\mu^2} - \frac{3\mu^4}{R}\right) + \mathcal{L}_{matter}. \tag{13}$$

This corresponds to

$$R_0 = 3\mu^2$$

and

$$M = M_{Pl} \Big/ \sqrt{2}\,,$$

and the power law for late time expansion would be the same as in the original CDTT model, $a(t) \propto t^2$.

4.2 Another model that satisfies the criterion (11) is

$$\mathcal{L} = \frac{M^2}{2}\sqrt{-g}\left(R - 15\frac{\mu^4}{R} + 25\frac{\mu^6}{R^2}\right) + \mathcal{L}_{matter}. \tag{14}$$

This yields

$$R_0 = 5\mu^2$$

and

$$M = M_{Pl}\sqrt{\frac{5}{6}}.$$

The R^{-2} term accelerates the power law expansion at late times to $a(t) \propto t^{3.75}$.

5 Conclusions

The problem of existence of a weak field expansion and the Newtonian limit is more intricate in singular NLG models than in regular NLG models, but can apparently be solved.
Models can in particular be chosen to satisfy the constraint (11) to ensure consistency of the weak field expansion at length scales $\ll R_0^{-1/2}$.
Two minimal extensions of the CDTT model which satisfy this constraint are given in Eqs. (13) and (14).

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